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STABLE STRUCTURES ON MANIFOLDS: I
HOMEOMORPHISMS OF S^n

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1. Introduction

If S is a locally flat $n-1$ sphere in the n-sphere S^n, then it is proved in [1] and [2] that the closures of the complementary domains of S are closed n-cells. On the other hand, if S_1 and S_2 are disjoint locally flat $n-1$ spheres in S^n, it is suspected but not known that the closure of the region between them is homeomorphic to $S^{n-1} \times [0, 1]$. The positive solution of this *annulus conjecture* would provide a powerful tool for the study of topological manifolds.

In this series of papers a machinery is constructed which permits the solution of some problems previously thought to be dependent on the annulus conjecture. The first paper considers the homeomorphisms of the n-sphere, and sets the stage for the development of the machinery in the following paper.

2. Definitions

The set of points $\{(x_1, \cdots, x_{n+1}): \sum x_i^2 = 1\}$ in euclidean $n+1$ space R^{n+1} will be denoted by S^n. S^n and any space homeomorphic to S^n will be called an *n-sphere*. Identifying R^n with $R^n \times 0 \subset R^{n+1}$ permits us to think of S^{n-1} as a subset of S^n, and write $S^{n-1} \subset S^n$.

$H(S^n)$ will denote the group of homeomorphisms of S^n, and $H^+(S^n)$ the subgroup of orientation preserving homeomorphisms. If A is a subset of S^n, $H(S^n, A)$ will denote the group of homeomorphisms of S^n which carry A onto itself.

An $n-1$ manifold M^{n-1} in an n-manifold M^n will be called *locally flat* if each point of M^{n-1} has a neighborhood U in M^n such that the pair $(U, U \cap M^{n-1})$ is topologically equivalent to (R^n, R^{n-1}). It is shown in [2] that a locally flat closed submanifold M^{n-1} which is two-sided in M^n has a neighborhood in M^n homeomorphic to $M^{n-1} \times [-1, 1]$, in which M^{n-1} appears as $M^{n-1} \times 0$. An embedding $f: M^{n-1} \rightarrow M^n$ is called locally flat

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if $f(M^{n-1})$ is locally flat in M^n.

The set of all locally flat embeddings $f: S^{n-1} \to S^n$ will be denoted by $\mathrm{Hom}(S^{n-1}, S^n)$. Recalling that $S^{n-1} \subset S^n$, we have the following relations between the elements of $H(S^n)$ and $\mathrm{Hom}(S^{n-1}, S^n)$.

(i) If $h \in H(S^n)$, then $h/S^{n-1} \in \mathrm{Hom}(S^{n-1}, S^n)$.

(ii) If $f \in \mathrm{Hom}(S^{n-1}, S^n)$, there is an $h \in H(S^n)$ such that $h/S^{n-1} = f$, according to [1] and [2].

(iii) If $f \in \mathrm{Hom}(S^{n-1}, S^n)$ and $h \in H(S^n)$, then $hf \in \mathrm{Hom}(S^{n-1}, S^n)$, and this action of $H(S^n)$ on $\mathrm{Hom}(S^{n-1}, S^n)$ is transitive by [1] and [2].

3. Annular equivalence of embeddings of S^{n-1} in S^n

The paper will proceed by first studying $\mathrm{Hom}(S^{n-1}, S^n)$ and then using the information obtained to study $H(S^n)$.

Let f_0 and f_1 be elements of $\mathrm{Hom}(S^{n-1}, S^n)$. If there is an *embedding* $F: S^{n-1} \times [0, 1] \to S^n$ such that, for all $x \in S^{n-1}$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, then F will be called a *strict annular equivalence* between f_0 and f_1, and we write

$$f_0 \underset{A}{\sim} f_1 .$$

Strict annular equivalence is not an equivalence relation, so we define annular equivalence as follows. Elements f and f' of $\mathrm{Hom}(S^{n-1}, S^n)$ will be called *annularly equivalent*, written

$$f \underset{a}{\sim} f' ,$$

if there is a finite sequence of elements $f = f_0, f_1, \cdots, f_k = f'$ of $\mathrm{Hom}(S^{n-1}, S^n)$ such that $f_i \underset{A}{\sim} f_{i+1}$. Strict annular equivalence is symmetric, hence so is annular equivalence. Annular equivalence is by construction transitive. Since a locally flat $n-1$ sphere in S^n has a neighborhood homeomorphic to $S^{n-1} \times [-1, 1]$, annular equivalence is reflexive. Hence annular equivalence is an equivalence relation.

LEMMA 3.1. *Let f be an element of* $\mathrm{Hom}(S^{n-1}, S^n)$ *and U an open set in S^n. Then there is an element f' of* $\mathrm{Hom}(S^{n-1}, S^n)$ *such that $f'(S^{n-1}) \subset U$ and $f \underset{A}{\sim} f'$.*

If C and D are the complementary domains of $f(S^{n-1})$ in S^n, then their closures, $\bar{C}$ and $\bar{D}$, are closed n-cells. U must meet either C or D, say C. Let F be a homeomorphism of the unit n-cell in R^n onto $\bar{C}$. It is evident that F may be chosen so that

(i) $F/S^{n-1} = f$.

(ii) F(origin) lies in $U \cap C$.

Then there is a $t > 0$ such that the image under F of the $n-1$ sphere

S_t of radius t and center at the origin lies in U. If p is the radial projection of S^{n-1} onto S_t, then $f' = Fp$ satisfies the required conditions.

Let f and f' be elements of Hom (S^{n-1}, S^n) such that $f(S^{n-1})$ and $f'(S^{n-1})$ are disjoint, and let R be the closed region of S^n between these image spheres. Choose an orientation for S^{n-1} and consider the induced orientations on $f(S^{n-1})$ and $f'(S^{n-1})$. If it is possible to orient R so that its boundary is $f(S^{n-1}) - f'(S^{n-1})$, we will say that f and f' have *similar orientations*. This is clearly independent of the original orientation chosen for S^{n-1}. If $f \underset{A}{\simeq} f'$, for example, then f and f' must have similar orientations. If f and f' do not have similar orientations, then they have *opposite orientations*.

It seems natural to inquire how much the relation of annular equivalence generalizes that of strict annular equivalence. Suppose, for example, that f and f' are annularly equivalent embeddings of S^{n-1} in S^n which look like they might be related by a strict annular equivalence (i.e., their images are disjoint, and they have similar orientations). Are they so related? The remainder of this section considers such questions.

LEMMA 3.2. *Let $f_1 \underset{A}{\simeq} f_2 \underset{A}{\simeq} f_3$ be disjoint embeddings such that one of the image spheres separates the other two. Then $f_1 \underset{A}{\simeq} f_3$.*

First suppose that $f_2(S^{n-1})$ separates $f_1(S^{n-1})$ from $f_3(S^{n-1})$. Let F_{12} be the strict annular equivalence of f_1 with f_2 and F_{23} the strict annular equivalence of f_2 with f_3. Define F_{13}: $S^{n-1} \times [0, 1] \to S^n$ by

$$\begin{aligned} F_{13}(x, t) &= F_{12}(x, 2t) && \text{for } 0 \leqq t \leqq 1/2 \\ &= F_{23}(x, 2t-1) && \text{for } 1/2 \leqq t \leqq 1\ . \end{aligned}$$

Then F_{13} is a strict annular equivalence of f_1 with f_3.

Next assume that $f_3(S^{n-1})$ separates $f_1(S^{n-1})$ from $f_2(S^{n-1})$, and that F_{12} and F_{23} are the given strict annular equivalences. The locally flat spheres $f_2(S^{n-1})$ and $f_3(S^{n-1})$ have neighborhoods in S^n homeomorphic to their product with an interval. Let E: $S^{n-1} \times [-1, 0] \to S^n$ be an embedding such that $E(x, 0) = f_2(x)$, and such that $E(S^{n-1} \times -1)$ is separated from $f_3(S^{n-1})$ by $f_2(S^{n-1})$. Let G: $S^{n-1} \times [1, 2] \to S^n$ be an embedding such that $G(x, 1) = f_3(x)$ and $G(S^{n-1} \times 2)$ lies between $f_3(S^{n-1})$ and $f_1(S^{n-1})$. Then define F^*: $S^{n-1} \times [-1, 2] \to S^n$ by putting E, F_{23} and G together. Using F^*, define a homeomorphism h of S^n which is the identity on $S^n -$ Image (F^*) and which takes $F^*(x, 0)$ onto $F^*(x, 1)$. Then $F_{13} = hF_{12}$ is a homeomorphism of $S^{n-1} \times [0, 1]$ into S^n.

$$F_{13}(x, 0) = hF_{12}(x, 0) = hf_1(x) = f_1(x)\ ,$$

since $f_1(S^{n-1})$ lies in $S^n -$ Image (F^*). And

$$F_{13}(x, 1) = hF_{12}(x, 1) = hf_2(x) = hF_{23}(x, 0) = hF^*(x, 0)$$
$$= F^*(x, 1) = F_{23}(x, 1) = f_3(x) \,.$$

Hence F_{13} is a strict annular equivalence between f_1 and f_3.

Proceed as above if $f_1(S^{n-1})$ is the separating sphere.

REMARK. If $f_1 \simeq_A f_2 \simeq_A f_3$ are disjoint embeddings, none of whose images separate the other two, then one can not possibly have $f_1 \simeq_A f_3$ because f_1 and f_3 will have opposite orientations.

LEMMA 3.3. *Let $f_1 \simeq_A f_2 \simeq_A f_3 \simeq_A f_4$, with all images disjoint and no image sphere separating the remaining image spheres. Then $f_1 \simeq_A f_4$.*

Each $f_i(S^{n-1})$ bounds a closed n-cell in S^n disjoint from the three other similar n-cells. Hence there is a homeomorphism h of S^n such that $hf_i(S^{n-1})$ is small for each i. Using Lemma 3.1 and the smallness of the $hf_i(S^{n-1})$, we can find an element $g \in \mathrm{Hom}\,(S^{n-1}, S^n)$ such that $g \simeq_A hf_2$ and such that $g(S^{n-1})$ contains $hf_1(S^{n-1})$ and $hf_3(S^{n-1})$ in one complementary domain and $hf_2(S^{n-1})$ and $hf_4(S^{n-1})$ in the other. Then Lemma 3.2 is used to show that

(i) $hf_1 \simeq_A hf_2 \simeq_A g$ implies $hf_1 \simeq_A g$,

(ii) $g \simeq_A hf_2 \simeq_A hf_3$ implies $g \simeq_A hf_3$,

(iii) $g \simeq_A hf_3 \simeq_A hf_4$ implies $g \simeq_A hf_4$,

(iv) $hf_1 \simeq_A g \simeq_A hf_4$ implies $hf_1 \simeq_A hf_4$.

But $hf_1 \simeq_A hf_4$ implies $f_1 \simeq_A f_4$.

LEMMA 3.4. *Let $f_1 \simeq_A f_2 \simeq_A f_3 \simeq_A f_4$ such that*

(i) all images are disjoint.

(ii) $f_1(S^{n-1})$ contains $f_2(S^{n-1})$ and $f_3(S^{n-1})$ in one complementary domain and $f_4(S^{n-1})$ in the other.

(iii) no other image sphere separates the remaining image spheres. Then there is a $g \in$ Hom (S^{n-1}, S^n) such that $g(S^{n-1})$ is a small sphere in the region between $f_1(S^{n-1})$ and $f_4(S^{n-1})$ and such that $f_1 \simeq_A g \simeq_A f_4$.

Let g be chosen so that $g(S^{n-1})$ is a small sphere in the region between $f_1(S^{n-1})$ and $f_4(S^{n-1})$ and such that $g \simeq_A f_1$, according to Lemma 3.1. Then $f_1(S^{n-1})$ separates $g(S^{n-1})$ from $f_2(S^{n-1})$ and $g \simeq_A f_1 \simeq_A f_2$. Hence by Lemma 3.2, $g \simeq_A f_2$. But g, f_2, f_3, f_4 satisfy the hypothesis of Lemma 3.3. Hence $g \simeq_A f_4$, as desired. Note that f_1 and f_4 have opposite orientations and therefore can not possibly be related by a strict annular equivalence.

Lemmas 3.2, 3.3 and 3.4 provide the tools for the proof of the following theorem, which gives the relation between annular equivalence and strict annular equivalence.

THEOREM 3.5. *Let f and f' be elements of* Hom (S^{n-1}, S^n) *such that $f \simeq_a f'$.*

(i) *If f and f' have disjoint images and similar orientations, then* $f \underset{A}{\simeq} f'$.

(ii) *There is a* $g \in \operatorname{Hom}(S^{n-1}, S^n)$ *such that* $f \underset{A}{\simeq} g \underset{A}{\simeq} f'$, *except in the case that* $f(S^{n-1}) = f'(S^{n-1})$ *and* $f^{-1}f'$ *is orientation reversing.*

(iii) *In this exceptional case, there are elements* $g, g' \in \operatorname{Hom}(S^{n-1}, S^n)$ *such that* $f \underset{A}{\simeq} g \underset{A}{\simeq} g' \underset{A}{\simeq} f'$.

PROOF OF (i). Let $f = f_0 \underset{A}{\simeq} f_1 \underset{A}{\simeq} \cdots \underset{A}{\simeq} f_k \underset{A}{\simeq} f_{k+1} = f'$. Assume that for no j does $f_j(S^{n-1})$ separate $f_{j-1}(S^{n-1})$ from $f_{j+1}(S^{n-1})$; otherwise, by Lemma 3.2 we could drop f_j from the chain and write $f_{j-1} \underset{A}{\simeq} f_{j+1}$. Then for each $j = 1, \cdots, k$, $f_j(S^{n-1})$ has a distinguished complementary domain D_j which contains neither $f_{j-1}(S^{n-1})$ nor $f_{j+1}(S^{n-1})$. For each $j = 1, \cdots, k$, let U_j be a small open set in D_j, chosen so that $U_i \cap U_j = \varnothing$ for $i \neq j$, and $U_j \cap f(S^{n-1}) = \varnothing = U_j \cap f'(S^{n-1})$ for all j. For each such j, use Lemma 3.1 to obtain an element $f_j^* \in \operatorname{Hom}(S^{n-1}, S^n)$ such that $f_j^*(S^{n-1}) \subset U_j$ and such that $f_j^* \underset{A}{\simeq} f_j$. An application of Lemma 3.2 yields $f \underset{A}{\simeq} f_1^*$ and $f_k^* \underset{A}{\simeq} f'$. A double application of Lemma 3.2 yields $f_j^* \underset{A}{\simeq} f_{j+1}^*$ for $j = 1, \cdots, k-1$. Thus

$$f = f_0 \underset{A}{\simeq} f_1^* \underset{A}{\simeq} \cdots \underset{A}{\simeq} f_k^* \underset{A}{\simeq} f_{k+1} = f' ,$$

with all images disjoint and $f_j^*(S^{n-1})$ small for $j = 1, \cdots, k$. Since a small sphere cannot separate the remaining spheres, Lemma 3.3 may be applied several times to the interior of this chain until we end up with either $f \underset{A}{\simeq} f'$, or $f \underset{A}{\simeq} f_1^* \underset{A}{\simeq} f'$, or $f \underset{A}{\simeq} f_1^* \underset{A}{\simeq} f_k^* \underset{A}{\simeq} f'$ or $f \underset{A}{\simeq} f_1^* \underset{A}{\simeq} f_2^* \underset{A}{\simeq} f_k^* \underset{A}{\simeq} f'$. Lemmas 3.2, 3.3 and 3.4 reduce the chains of length 4 and 5 to chains of length 2 or 3. A chain of length 3 between f and f' either reduces to a chain of length 2 by Lemma 3.2 or to the conclusion that f and f' have opposite orientations, hence in the present case, to a chain of length 2. Invariably, then, we end up with $f \underset{A}{\simeq} f'$, completing the argument for (i).

PROOF OF (ii). If $f(S^{n-1}) = f'(S^{n-1})$ and $f^{-1}f'$ is orientation preserving, let g be a locally flat embedding of S^{n-1} in one of the complementary domains of $f(S^{n-1})$, chosen so that $g \underset{A}{\simeq} f$, according to Lemma 3.1. Then $g \underset{A}{\simeq} f \underset{a}{\simeq} f'$ implies $g \underset{a}{\simeq} f'$. But g and f' satisfy the conditions of (i), hence $g \underset{A}{\simeq} f'$.

If $f(S^{n-1}) \neq f'(S^{n-1})$, let x be a point of S^n which lies on $f'(S^{n-1})$ but not on $f(S^{n-1})$. Let U be an open neighborhood of x in S^n which misses $f(S^{n-1})$ and which is chosen, according to the local flatness of $f'(S^{n-1})$, so that the pair $(U, U \cap f'(S^{n-1}))$ is homeomorphic to the pair (R^n, R^{n-1}). Orienting S^{n-1} induces orientations of $f(S^{n-1})$ and $f'(S^{n-1})$, which in turn induce orientations on their complementary domains. Let U' be that component of $U - f'(S^{n-1})$ whose orientation as induced by the orientation of $f'(S^{n-1})$ coincides with the orientation induced by the orientation of $f(S^{n-1})$. Now let g be an embedding of S^{n-1} in U' such that $g \underset{A}{\simeq} f$, according to Lemma 3.1. Then g and f' have disjoint images and are similarly oriented. Furthermore $g \underset{A}{\simeq} f \underset{a}{\simeq} f'$ implies $g \underset{a}{\simeq} f'$. Then by (i), $g \underset{A}{\simeq} f'$.

PROOF OF (iii). If $f(S^{n-1}) = f'(S^{n-1})$ and $f^{-1}f'$ is orientation reversing, let g be an embedding of S^{n-1} in one of the complementary domains of $f(S^{n-1})$, chosen so that $f \underset{A}{\simeq} g$. Let g' be a small embedding of S^{n-1} in the region between $f(S^{n-1})$ and $g(S^{n-1})$, chosen so that $g \underset{A}{\simeq} g'$. Then $f \underset{a}{\simeq} g'$, but f and g' have opposite orientations. Therefore f' and g' have similar orientations, and $f' \underset{a}{\simeq} g'$. By (i), $f' \underset{A}{\simeq} g'$, which proves (iii).

4. Stable homeomorphisms

Let h be a homeomorphism of S^n onto itself. If there is a non-empty open set $U \subset S^n$ such that $h/U = 1$, we will say that h is *somewhere the identity*. Then $SH(S^n)$, the group of *stable homeomorphisms* of S^n, will consist of products of homeomorphisms, each of which is somewhere the identity.

$SH(S^n)$ is the intersection of all non-trivial normal subgroups of $H(S^n)$ and is, furthermore, simple [3]. Then $SH(S^n)$ must be arcwise connected in the compact-open topology. Hence every stable homeomorphism is isotopic to the identity through stable homeomorphisms. If $h \in H(S^n)$ agrees with $h_0 \in SH(S^n)$ on the non-empty open set U, then $h_0^{-1}h$ is the identity on U, and hence in $SH(S^n)$. Then also $h \in SH(S^n)$. Hence any homeomorphism of S^n which agrees with a stable homeomorphism on a non-empty open set is itself stable.

If S is a locally flat $n-1$ sphere in S^n and h is a homeomorphism of S^n, whose restriction to S is the identity and which does not interchange the complementary domains of S, then h is stable. For if the complementary domains of S in S^n are C and D, let h_1 be the homeomorphism of S^n which is the identity on C and agrees with h on D. Then $h = (hh_1^{-1})h_1$, and h_1 is the identity on C and hh_1^{-1} is the identity on D, from which it follows that h is stable.

The principal property of stable homeomorphisms is expressed in Theorem 7.1, from which it follows in the Corollary that any stable homeomorphism of S^n can be written as the product of just *two* homeomorphisms, each of which is somewhere the identity.

5. Stable equivalence of embeddings of S^{n-1} in S^n

Let f_1 and f_2 be elements of Hom (S^{n-1}, S^n). Then by [1] and [2] there is a homeomorphism h of S^n such that $hf_1 = f_2$, and if desired, h may be chosen to be orientation preserving. If there is an $h \in SH(S^n)$ such that $hf_1 = f_2$, then we say that f_1 and f_2 are *stably equivalent* and write $f_1 \underset{s}{\simeq} f_2$. Stable equivalence is an equivalence relation, and $\mathrm{Hom}(S^{n-1}, S^n)$ divides up into stable equivalence classes.

Let $f_1 \underset{s}{\simeq} f_2$, and choose $h_0 \in SH(S^n)$ so that $h_0 f_1 = f_2$. If $h \in H(S^n)$, then $hh_0h^{-1} \in SH(S^n)$ by normality. Then $(hh_0h^{-1})hf_1 = hf_2$ yields $hf_1 \underset{s}{\simeq} hf_2$. Thus $H(S^n)$ acts on $\mathrm{Hom}(S^{n-1}, S^n)$ by permuting the stable equivalence classes.

LEMMA 5.1, *Let f_1 and f_2 be stably equivalent elements of* $\mathrm{Hom}(S^{n-1}, S^n)$, *and h an orientation preserving homeomorphism of S^n such that $hf_1 = f_2$. Then $h \in SH(S^n)$.*

Choose $h_0 \in SH(S^n)$ such that $h_0 f_1 = f_2$. Then $h_0^{-1}hf_1 = f_1$. Hence $h_0^{-1}h / f_1(S^{n-1})$ is the identity. Since $h_0^{-1}h$ is orientation preserving, it does not interchange the complementary domains of $f_1(S^{n-1})$, and is therefore stable by § 4. Then h must also be stable.

COROLLARY 1. *If an element of $H^+(S^n)$ leaves one stable equivalence class of* $\mathrm{Hom}(S^{n-1}, S^n)$ *fixed, it is an element of $SH(S^n)$, and therefore leaves all stable equivalence classes fixed.*

COROLLARY 2. *$H^+(S^n)/SH(S^n)$ acts transitively and regularly on the stable equivalence classes of* $\mathrm{Hom}(S^{n-1}, S^n)$, *and is therefore in one-one correspondence with the set of these stable equivalence classes.*

THEOREM 5.2. *Let f and f' be elements of* $\mathrm{Hom}(S^{n-1}, S^n)$ *such that $f \underset{A}{\simeq} f'$. Then there is a stable homeomorphism h of S^n such that $f = hf'$.*

Let $F: S^{n-1} \times [0, 1] \rightarrow S^n$ be a strict annular equivalence between f and f'. Since $f(S^{n-1})$ and $f'(S^{n-1})$ are locally flat, F may be extended to an embedding $F^*: S^{n-1} \times [-1, 2] \rightarrow S^n$. Let h be a homeomorphism of S^n which takes $F^*(x, 1)$ onto $F^*(x, 0)$ and which is the identity on $S^n - \mathrm{Image}(F^*)$. Then h is stable and

$$hf'(x) = hF(x, 1) = hF^*(x, 1) = F^*(x, 0) = F(x, 0) = f(x) .$$

COROLLARY. *If $f, f' \in$* $\mathrm{Hom}(S^{n-1}, S^n)$ *and $f \underset{a}{\simeq} f'$, then $f \underset{s}{\simeq} f'$.*

THEOREM 5.3. *Let h be a stable homeomorphism of S^n whose restriction to the non-empty open set U is the identity. If $f \in$* $\mathrm{Hom}(S^{n-1}, S^n)$, *then $f \underset{a}{\simeq} hf$.*

Let f' be an element of $\mathrm{Hom}(S^{n-1}, S^n)$ such that $f'(S^{n-1}) \subset U$ and $f' \underset{A}{\simeq} f$, according to Lemma 3.1. Then $hf' \underset{A}{\simeq} hf$. But $hf' = f'$. Therefore $f \underset{A}{\simeq} f' = hf' \underset{A}{\simeq} hf$. Hence $f \underset{a}{\simeq} hf$.

COROLLARY. *If* $f, f' \in \mathrm{Hom}(S^{n-1}, S^n)$ *and* $f \underset{s}{\simeq} f'$, *then* $f \underset{a}{\simeq} f'$.

Choose a stable homeomorphism h such that $f' = hf$. Write h as a product $h_k h_{k-1} \cdots h_2 h_1$ of stable homeomorphisms each of whose restriction to some (variable) non-empty open set is the identity. Then by Theorem 5.3,

$$f \underset{a}{\simeq} h_1 f \underset{a}{\simeq} h_2 h_1 f \underset{a}{\simeq} \cdots \underset{a}{\simeq} h_k h_{k-1} \cdots h_2 h_1 f = f' ,$$

hence $f \underset{a}{\simeq} f'$.

Combining these two corollaries, we get

THEOREM 5.4. *Two elements of* $\mathrm{Hom}(S^{n-1}, S^n)$ *are stably equivalent if and only if they are annularly equivalent.*

We will generally prefer the name *stable equivalence* for this relation, except when specifically referring to the results of § 3.

6. Approximation of spheres

THEOREM 6.1. *Let* $F: S^{n-1} \times [0, 1] \to S^n$ *be a (not necessarily locally flat) embedding, and* g *an arbitrary element of* $\mathrm{Hom}(S^{n-1}, S^n)$. *Then there is an element* g' *of* $\mathrm{Hom}(S^{n-1}, S^n)$ *such that*

(i) g' *is stably equivalent to* g,

(ii) $g'(S^{n-1}) \subset F(S^{n-1} \times (0, 1))$ *and separates* $F(S^{n-1} \times 0)$ *from* $F(S^{n-1} \times 1)$.

Using [1] and [2], first construct a stable homeomorphism h_1 of S^n which is the identity in a neighborhood of $F(S^{n-1} \times 1/2)$ and which shrinks $F(S^{n-1} \times 0)$ and $F(S^{n-1} \times 1)$ very small. Then let h_2 be a stable homeomorphism which slides $h_1 F(S^{n-1} \times 0)$ into one complementary domain of $g(S^{n-1})$ and $h_1 F(S^{n-1} \times 1)$ into the other. Finally let $g' = h_1^{-1} h_2^{-1} g$. Since h_1 and h_2 are stable, g' is stably equivalent to g. And since $g(S^{n-1})$ separates $h_2 h_1 F(S^{n-1} \times 0)$ from $h_2 h_1 F(S^{n-1} \times 1)$, $g'(S^{n-1})$ separates $F(S^{n-1} \times 0)$ from $F(S^{n-1} \times 1)$.

REMARK. Note that we can arrange the 'orientation' of g' by sliding $h_1 F(S^{n-1} \times 0)$ into the appropriate complementary domain of $g(S^{n-1})$. If $g(S^{n-1})$ does not meet $F(S^{n-1} \times (0, 1))$, we can therefore arrange that g and g' have similar orientations. In such a case it would then follow from Theorem 3.5 that $g \underset{A}{\simeq} g'$.

7. The structure of stable homeomorphisms

THEOREM 7.1. *Let* h *be a stable homeomorphism of* S^n *and* E_1, E_2 *closed* n*-cells with locally flat boundaries* S_1, S_2. *If* $E_1 \cup hE_1$ *is disjoint from* E_2, *then there is a stable homeomorphism* h' *of* S^n *which agrees with* h *on* E_1 *and whose restriction to* E_2 *is the identity.*

Let f be any homeomorphism of S^{n-1} onto S_1. Use Theorem 6.1 to find an element f' of Hom (S^{n-1}, S^n) which is stably equivalent to f and whose image separates $E_1 \cup hE_1$ from E_2. Use the Remark following Theorem 6.1 to insure that f and f' have similar orientations, so that $f \underset{A}{\sim} f'$. Let E_2' denote the closed n-cell bounded by $f'(S^{n-1})$ which contains E_2. Since h is stable, f' is also stably equivalent to hf. Note that f' and hf have disjoint images and similar orientations. Then by Theorem 3.5, $hf \underset{A}{\sim} f'$.

Now let F be a strict annular equivalence between f and f' and F^* a strict annular equivalence between hf and f'. Define a homeomorphism h' from $F(S^{n-1} \times [0, 1])$ onto $F^*(S^{n-1} \times [0, 1])$ by taking $F(x, t)$ onto $F^*(x, t)$. Then h'/S_1 takes $F(x, 0)$ onto $F^*(x, 0)$, i.e., takes $f(x)$ onto $hf(x)$, i.e., coincides with h. And $h'/f'(S^{n-1})$ takes $F(x, 1)$ onto $F^*(x, 1)$, i.e., takes $f'(x)$ onto $f'(x)$, i.e., is the identity. Finally, extend h' over S^n by taking E_1 onto hE_1 by h and E_2' onto E_2' by the identity. Then h' satisfies the conditions of the theorem.

COROLLARY. *Any stable homeomorphism of S^n can be written as the product of two homeomorphisms, each of which is somewhere the identity.*

For if h is a stable homeomorphism of S^n, one uses the above theorem to find another stable homeomorphism h' which agrees with h on a small n-cell E_1 and restricts to the identity on some other n-cell E_2, and then writes $h = h'(h'^{-1}h)$.

8. The suspension homomorphism

Let S be a locally flat $n-1$ sphere in S^n, $H(S)$ the group of homeomorphisms of S and g an element of $H(S)$. Let h be any extension of g over S^n which does not interchange the complementary domains of S. The extension h is certainly not unique, but if h' is any other extension, then $h^{-1}h'$ is a homeomorphism of S^n which is the identity on S, and does not interchange the complementary domains of S. By § 4, $h^{-1}h'$ must be stable. Thus the coset $h \cdot SH(S^n)$ *is* uniquely determined by g. The correspondence $g \to h \cdot SH(S^n)$ defines a homomorphism

$$\Sigma\colon H(S) \to H(S^n)/SH(S^n)\ ,$$

called the *suspension homomorphism*.

A coset of $SH(S^n)$ is in the image of Σ if and only if that coset contains a homeomorphism which is invariant on S and does not interchange the complementary domains of S. Note however that $SH(S^n)$ contains a homeomorphism which is invariant on S and *does* interchange the complementary domains of S. For if h is a homeomorphism of S^n which takes S onto S^{n-1}, and t a rigid deformation of S^n which interchanges the northern and southern hemispheres, then $h^{-1}th$ is invariant on S and interchanges the complementary domains of S. One easily sees that t is stable, therefore $h^{-1}th$ is also stable. Hence a coset of $SH(S^n)$ is in the image of Σ if and only if that coset contains a homeomorphism of S^n which is invariant on S. Letting $H(S^n, S)$ denote the group of homeomorphisms of S^n which are invariant on S, we have

THEOREM 8.1. *The image of* Σ *is* $\big(H(S^n, S) \cdot SH(S^n)\big)/SH(S^n)$.

DEFINITION 8.2. The homeomorphism f of a space X onto itself is said to be *weakly isotopic* to the homeomorphism f' of X if there is a homeomorphism $F: X \times [0, 1] \to X \times [0, 1]$ such that $F(x, 0) = (f(x), 0)$ and $F(x, 1) = (f'(x), 1)$.

$HI(S)$ will denote the normal subgroup of $H(S)$ consisting of those homeomorphisms weakly isotopic to the identity.

THEOREM 8.3. *The kernel of* Σ *is* $HI(S)$.

Suppose first that g is a homeomorphism of S which is weakly isotopic to the identity. Let G be a homeomorphism of $S \times [0, 1]$ onto itself such that $G(x, 0) = (g(x), 0)$ and $G(x, 1) = (x, 1)$. Let $F: S \times [0, 1] \to S^n$ be a locally flat embedding such that $F(x, 0) = x$. Define the homeomorphism h of $F\big(S \times [0,1]\big)$ onto itself by $h = FGF^{-1}$. Then $h/S = g$ and $h/F(S \times 1)$ is the identity. Extend h to a homeomorphism of S^n which is the identity on one of the complementary domains of $F(S \times 1)$. Then h is stable, hence g is in the kernel of Σ.

Now let g be in the kernel of Σ and let E_1 and E_1' be the closed n-cells in S^n bounded by S. Let $F: S \times [0, 1] \to S^n$ be a locally flat embedding such that $F(x, 0) = x$ and $F(S \times 1) \subset E_1'$. Let E_2 be the closed n-cell bounded by $F(S \times 1)$ and contained in E_1'. Let h be an extension of g over S^n which does not interchange E_1 and E_1'. Since g is in the kernel of Σ, h must be stable. Apply Theorem 7.1 to h, E_1 and E_2 to obtain a stable homeomorphism h' which agrees with h on E_1 and the identity on E_2. Then $h'F\big(S \times [0,1]\big) = F\big(S \times [0,1]\big)$ and $F^{-1}h'F: S \times [0,1] \to S \times [0,1]$ is a weak isotopy of g with the identity.

THEOREM 8.4. *The suspension homomorphism* Σ *induces an isomorphism*

$$\big(H(S)\big)/\big(HI(S)\big) \approx \big(H(S^n, S) \cdot SH(S^n)\big)/\big(SH(S^n)\big) ,$$

where S *is a locally flat* $n - 1$ *sphere in* S^n.

It will be seen later that $HI(S^3) = H^+(S^3)$, from which we obtain

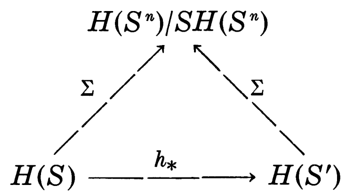
THEOREM 8.5. $(H(S^4, S) \cdot SH(S^4))/(SH(S^4)) \approx Z_2$, *where* S *is a locally flat* 3-*sphere in* S^4. *In particular, if* h *is an orientation preserving* *homeomorphism of S^4 which is invariant on some locally flat 3-sphere in S^4, then h is stable.*

DEFINITION 8.6. The locally flat $n-1$ spheres S and S' in S^n will be called *stably equivalent* if there is a stable homeomorphism h of S^n such that $h(S) = S'$. Note that h induces an isomorphism $h_*\colon H(S) \to H(S')$ defined by $h_*(g) = hgh^{-1}/S'$.

THEOREM 8.7. *Under the conditions of the above definition,*

$$H(S^n, S) \cdot SH(S^n) = H(S^n, S') \cdot SH(S^n) ,$$

and the following diagram is commutative.



Let g be a homeomorphism of S and $\bar{g}$ an extension of g over S^n which does not interchange the complementary domains of S. Then $h\bar{g}h^{-1}$ is an extension of $h_*(g)$ over S^n which does not interchange the complementary domains of S'. Then

$$\Sigma\big(h_*(g)\big) = (h\bar{g}h^{-1}) \cdot SH(S^n) = \bar{g} \cdot SH(S^n) = \Sigma(h) ,$$

since $SH(S^n)$ is normal in $H(S^n)$. The diagram is therefore commutative, from which it follows that $H(S^n, S) \cdot SH(S^n) = H(S^n, S') \cdot SH(S^n)$.

9. The relation between stable homeomorphisms and the annulus conjecture

By the *annulus conjecture in dimension n*, we mean the following statement, denoted by A_n.

A_n. *If S_1 and S_2 are disjoint locally flat $n-1$ spheres in S^n, then the closure of the region between them is homeomorphic to $S^{n-1} \times [0, 1]$.*

LEMMA 9.1. *A_n is true if and only if any two (not necessarily disjoint) locally flat $n-1$ spheres in S^n are stably equivalent.*

If A_n is true and S_1 and S_2 are locally flat $n-1$ spheres in S^n, let h_1 be a stable homeomorphism such that $h_1(S_1)$ is disjoint from S_2. Applying A_n to $h_1(S_1)$ and S_2, one easily constructs a stable homeomorphism h_2 such that $h_2h_1(S_1) = S_2$. Hence S_1 and S_2 are stably equivalent.

If S_1 and S_2 are disjoint locally flat $n-1$ spheres in S^n which are stably equivalent, let h be a stable homeomorphism such that $h(S_1) = S_2$. Let f

be a homeomorphism of S^{n-1} onto S_1, and compare the elements f and hf of $\mathrm{Hom}(S^{n-1}, S^n)$. They are stably equivalent and have disjoint images, but may or may not have similar orientations. If they have opposite orientations, let h' be a stable homeomorphism of S^n which is invariant on S_2 and which interchanges the complementary domains of S_2 (see § 8), and compare f with $h'hf$ instead of with hf. In either case we obtain elements f_1 and f_2 of $\mathrm{Hom}(S^{n-1}, S^n)$ which are stably equivalent, have disjoint images S_1 and S_2 and have similar orientations. By Theorem 3.5, $f_1 \underset{A}{\simeq} f_2$, hence the closed region between S_1 and S_2 must be homeomorphic to $S^{n-1} \times [0, 1]$.

THEOREM 9.2. *A_n is true if and only if $H(S^n) = H(S^n, S^{n-1}) \cdot SH(S^n)$.*

If A_n is true and $h \in H(S^n)$, then S^{n-1} and $h^{-1}(S^{n-1})$ must be stably equivalent by Lemma 9.1. Let h_0 be a stable homeomorphism which carries S^{n-1} onto $h^{-1}(S^{n-1})$. Then writing $h = (hh_0)h_0^{-1}$ yields $H(S^n) = H(S^n, S^{n-1}) \cdot SH(S^n)$.

If $H(S^n) = H(S^n, S^{n-1}) \cdot SH(S^n)$ and S_1 and S_2 are two locally flat $n-1$ spheres, let g be a homeomorphism of S^n such that $g(S_1) = S^{n-1}$. Let h be a homeomorphism of S^n such that $h(S^{n-1}) = g(S_2)$, and, using the normality of $SH(S^n)$, write $h = h'h''$, with h' stable and h'' invariant on S^{n-1}. Then $g(S_2) = h'h''(S^{n-1}) = h'(S^{n-1})$, hence $g(S_2)$ is stably equivalent to S^{n-1}. Then S_2 is stably equivalent to $g^{-1}(S^{n-1}) = S_1$, hence by Lemma 9.1, A_n is true.

REMARK. Clearly Theorem 9.2 remains valid if we replace S^{n-1} by any locally flat $n-1$ sphere S in S^n. Then it follows from Theorem 8.1 that A_n is true if and only if the suspension homomorphism $\Sigma\colon H(S) \to H(S^n)/SH(S^n)$ is onto for every locally flat $n-1$ sphere S in S^n.

THEOREM 9.3. *$SH(S^n) = H^+(S^n)$ if and only if A_n is true and $HI(S^{n-1}) = H^+(S^{n-1})$.*

If $SH(S^n) = H^+(S^n)$, then

$$H(S^n) = H(S^n, S^{n-1}) \cdot H^+(S^n) = H(S^n, S^{n-1}) \cdot SH(S^n),$$

so by Theorem 9.2, A_n is true.

If $SH(S^n) = H^+(S^n)$, then $H(S^{n-1})/HI(S^{n-1}) \approx Z_2$ by Theorem 8.4. But $HI(S^{n-1}) \subset H^+(S^{n-1})$ and $H(S^{n-1})/H^+(S^{n-1}) \approx Z_2$. Therefore $HI(S^{n-1}) = H^+(S^{n-1})$.

If A_n is true, then $H(S^n) = H(S^n, S^{n-1}) \cdot SH(S^n)$ by Theorem 9.2, and if $HI(S^{n-1}) = H^+(S^{n-1})$, then $H(S^{n-1})/HI(S^{n-1}) \approx Z_2$. Then by Theorem 8.4, $H(S^n)/SH(S^n) \approx Z_2$. Hence, as above, $SH(S^n) = H^+(S^n)$.

THEOREM 9.4. *A_k is true for $k \leqq n$ if and only if $SH(S^k) = H^+(S^k)$ for $k \leqq n$.*

If A_k is true and $SH(S^{k-1}) = H^+(S^{k-1})$, then also $HI(S^{k-1}) = H^+(S^{k-1})$, for $SH(S^{k-1}) \subset HI(S^{k-1}) \subset H^+(S^{k-1})$. From Theorem 9.3 it follows that $SH(S^k) = H^+(S^k)$. Hence, by induction, if A_k is true for $k \leqq n$, then $SH(S^k) = H^+(S^k)$ for $k \leqq n$.

The other direction follows immediately from Theorem 9.3.

REMARKS. 1. In summary, the statement that $SH(S^n) = H^+(S^n)$ is stronger than the statement that the annulus conjecture is true in dimension n, but the statement that $SH(S^k) = H^+(S^k)$ for $k \leqq n$ is equivalent to the statement that the annulus conjecture is true in dimensions less than or equal to n.

2. It is shown in [5] that, if the closed region between two disjoint locally flat $k-1$ spheres in S^k can be triangulated for $k \leqq n$, then A_k is true for $k \leqq n$. Then by [6] and [7], A_k is true for $k \leqq 3$. Hence $SH(S^k) = H^+(S^k)$ for $k \leqq 3$.

10. Extension of homeomorphisms

THEOREM 10.1. *Let $S_1, \cdots, S_k$ be disjoint locally flat $n-1$ spheres in S^n, and g_i a homeomorphism of S_i onto itself for $i = 1, \cdots, k$. Then a necessary and sufficient condition for the existence of a homeomorphism h of S^n onto itself such that $h/S_i = g_i$ is that $\Sigma(g_1) = \cdots = \Sigma(g_k) \in H(S^n)/SH(S^n)$.*

If $k = 1$, the condition is vacuous and hence necessary. If $k > 1$, and such a homeomorphism h exists, then h cannot interchange the complementary domains of any S_i. Hence for each i, h lies in the coset $\Sigma(g_i)$, and the condition is therefore necessary.

If $k = 1$, h always exists by [1] and [2], and may be chosen so as not to interchange the complementary domains of S_1. Suppose then that $k > 1$ and that the theorem has already been proved for smaller values of k. Number the spheres so that S_k is an innermost sphere, i.e., so that one of its complementary domains contains none of the S_i. Let h' be a homeomorphism of S^n which extends g_i for $i = 1, \cdots, k-1$. If $k > 2$, then h' cannot interchange the complementary domains of any S_i; and if $k = 2$, we have remarked above that h' can be so chosen. Let E_k be the closed n-cell bounded by S_k which contains none of the other spheres. Then E_k and $h'(E_k)$ lie in the same component of $S^n - (S_1 \cup \cdots \cup S_{k-1})$. Let h'' be a stable homeomorphism of S^n which shrinks $h'(E_k)$ and slides it inside the interior of E_k, and which is the identity in a neighborhood of $S_1 \cup \cdots \cup S_{k-1}$.

Let f be any homeomorphism of S^{n-1} onto S_k. Then $g_k f$ and $h''h'f$ are elements of $\text{Hom}(S^{n-1}, S^n)$. If $\bar{g}_k$ is any extension of g_k to a homeomorphism of S^n which does not interchange the complementary domains of S_k, then $\bar{g}_k$ lies in the coset $\Sigma(g_k)$ of $SH(S^n)$. So does h', since all the $\Sigma(g_i)$ are equal. Since h'' is stable, so does $h''h'$. Then $h''h'\bar{g}_k^{-1}$ must be a stable homeomorphism, hence the embeddings $g_k f = \bar{g}_k f$ and $h''h'f$ are stably equivalent. Since their images are disjoint and $h''h'(E_k)$ lies inside the interior of $\bar{g}_k(E_k) = E_k$, $g_k f$ and $h''h'f$ have similar orientations. Then by Theorem 3.5, $g_k f \underset{A}{\simeq} h''h'f$. If $F: S^{n-1} \times [0, 1] \to S^n$ is such a strict annular equivalence, extend F to an embedding of $S^{n-1} \times [-1, 2]$ in S^n, whose image contains none of the $n-1$ spheres other than S_k. Using this embedding, define a homeomorphism h''' of S^n which takes $F(x, 1)$ onto $F(x, 0)$ and is the identity outside the image of $S^{n-1} \times [-1, 2]$.

Then $h = h'''h''h'$ satisfies the requirements of the theorem. For if $y \in S_k$, write $y = f(x)$, $x \in S^{n-1}$. Then

$$h(y) = h'''\big(h''h'f(x)\big) = h'''\big(F(x, 1)\big) = F(x, 0) = g_k f(x) = g_k(y) .$$

If $i < k$, then $h/S_i = h'/S_i = g_i$, and the proof is completed.

THEOREM 10.2. *Let $S_1, \cdots, S_k$ be disjoint locally flat $n-1$ spheres in S^n which are stably equivalent to one another. If g is a homeomorphism of S_1 onto itself, then there is a homeomorphism h of S^n such that $h/S_1 = g$ and $h(S_i) = S_i$ for $i = 2, \cdots, k$.*

By Theorem 8.7, $\Sigma(H(S_1)) = \cdots = \Sigma(H(S_k))$. Hence there are homeomorphisms g_i of S_i, $i = 2, \cdots, k$, such that $\Sigma(g_i) = \Sigma(g)$. Then apply Theorem 10.1.

THEOREM 10.3. *Let $S_1, \cdots, S_k$ be disjoint locally flat $n-1$ spheres in S^n which are stably equivalent to one another. Suppose that none of the spheres separates the remaining spheres, and let R be the closed region of S^n bounded by all of the spheres. Then the topological type of R is independent of the particular choice of the S_i, and is referred to as a sphere with k holes.*

By a *round* $n-1$ sphere in S^n, we mean an $n-1$ sphere which is the intersection of S^n with an n-dimensional hyperplane in R^{n+1}. We may assume, by [1] and [2], that S_1 is round and that $S_2, \cdots, S_k$ are very small. Surround $S_2, \cdots, S_k$ by round spheres $S_2', \cdots, S_k'$ in the obvious way, and let R' denote the closed region bounded by $S_1, S_2', \cdots, S_k'$. The topological type of R' is clearly independent of the particular choice of round spheres. Since S_1 is round, S_i is stably equivalent to round spheres and hence to S_i'. Then by Lemma 9.1 and Theorem 3.5, the closed region between S_i and S_i' is homeomorphic to $S^{n-1} \times [0,1]$. Hence R is obtained from R' by adding $k-1$ rims, each homeomorphic to $S^{n-1} \times [0, 1]$, which does not change the topological type of R'.

COROLLARY. *Let $S_1, \cdots, S_k$ be disjoint locally flat $n-1$ spheres in S^n which are stably equivalent to one another. Then there is a homeomorphism h of S^n such that $h(S_i)$ is round for each i.*

THEOREM 10.4. *Let $f_1, \cdots, f_k$ be homeomorphisms of S^n and $x_1, \cdots, x_k$ distinct points of S^n chosen so that $f_1(x_1), \cdots, f_k(x_k)$ are also distinct. Then a necessary and sufficient condition for the existence of a homeomorphism h of S^n which agrees with each f_i on a neighborhood of x_i is that $f_1, \cdots, f_k$ should all lie in the same coset of $SH(S^n)$ in $H(S^n)$.*

If such an h exists, then since f_i and h agree on an open set, they must lie in the same coset of $SH(S^n)$, so the condition is necessary.

Let E_i be a closed n-cell with round boundary S_i, which contains x_i in its interior, and which is chosen so small that not only are all the E_i disjoint but also all the $f_i(E_i)$ are disjoint. If h_{ij} is a stable homeomorphism taking S_i onto S_j, then, since f_i and f_j lie in the same coset of $SH(S^n)$, $f_j h_{ij} f_i^{-1}$ is a stable homeomorphism taking $f_i(S_i)$ onto $f_j(S_j)$. Hence the $f_i(S_i)$ are stably equivalent to one another and so, by Theorem 10.3, there is a homeomorphism h' of S^n which takes S_i onto $f_i(S_i)$ for each i. Then $h'^{-1}f_i$ is invariant on E_i for each i, and the several $h'^{-1}f_i$ all lie in the same coset of $SH(S^n)$. By Theorem 10.1 there is a homeomorphism h'' of S^n which agrees with $h'^{-1}f_i$ on S_i for each i. Then h'' may obviously be taken to agree with $h'^{-1}f_i$ on E_i for each i. Hence $h = h'h''$ agrees with f_i on E_i for each i, proving the theorem.

11. Stability of differentiable and piecewise linear homeomorphisms

THEOREM 11.1. *Let Σ^n be a differentiable manifold whose underlying space is S^n, and h an orientation preserving diffeomorphism of Σ^n onto itself. Then h is stable.*

Let E be a small differentiable closed n-cell in Σ^n. Then the inclusion $E \subset \Sigma^n$ and the map h/E: $E \to \Sigma^n$ are orientation preserving differentiable embeddings of E in Σ^n. Since E and hence $h(E)$ are small, let U be a proper open subset of Σ^n containing both E and $h(E)$. Then U inherits a differentiable structure from Σ^n. By [8], the two embeddings of E in U are equivalent in the sense that there is a diffeomorphism h' of U, which is the identity outside a compact set, such that $h'/E = h/E$. Then h' extends, *via* the identity, to a stable homeomorphism of Σ^n. Since h' and h agree on E, h must also be stable.

The same argument in piecewise linear topology, using [9] and [10] instead of [8], proves

THEOREM 11.2. *Let Σ^n be a combinatorial manifold whose underlying* *space is S^n, and h an orientation preserving piecewise linear homeomorphism of Σ^n onto itself. Then h is stable.*

12. Homeomorphisms of euclidean space

Let z denote the north pole of S^n and $p: S^n - z \to R^n$ the stereographic projection of $S^n - z \subset R^{n+1}$ onto $R^n \subset R^{n+1}$. Then to each homeomorphism h of (S^n, z) there corresponds the homeomorphism $p(h/S^n - z)p^{-1}$ of R^n. This correspondence defines an isomorphism

$$p^*: H(S^n, z) \to H(R^n)$$

between the group of homeomorphisms of S^n which leave the north pole fixed and the group of all homeomorphisms of R^n.

Let $SH(R^n)$, the group of stable homeomorphisms of R^n, be generated by homeomorphisms whose restriction to some variable non-empty open set is the identity. Then $SH(R^n)$ is the image under p^* of $SH(S^n, z)$, the group of stable homeomorphisms of S^n which leave the north pole fixed. $SH(R^n)$, unlike $SH(S^n)$, is not simple. The smallest non-trivial normal subgroup of $H(R^n)$ is $SH_0(R^n)$, the group of homeomorphisms which restrict to the identity outside some variable compact set.

THEOREM 12.1. $(H(R^n))/(SH(R^n)) \approx (H(S^n))/(SH(S^n))$.

For

$$\begin{aligned}(H(R^n))/(SH(R^n)) &\approx (H(S^n, z))/(SH(S^n, z)) \\ &\approx (H(S^n, z) \cdot SH(S^n))/\ (SH(S^n)) = (H(S^n))/(SH(S^n),\end{aligned}$$

because $SH(S^n)$ acts transitively on S^n.

The theorems of the preceding sections translate without difficulty to theorems about homeomorphisms of euclidean space. For the convenience of later reference, we restate Theorem 7.1 for R^n.

THEOREM 12.2. *Let h be a stable homeomorphism of R^n, and E a closed n-cell with locally flat boundary. Then there is a stable homeomorphism h' of R^n which agrees with h on E and which restricts to the identity outside some compact set.*

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